

# An equivariant Poincaré series of filtrations and monodromy zeta functions\*

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## Abstract

We define a new equivariant (with respect to a finite group  $G$  action) version of the Poincaré series of a multi-index filtration as an element of the power series ring  $\tilde{A}(G)[[t_1, \dots, t_r]]$  for a certain modification  $\tilde{A}(G)$  of the Burnside ring of the group  $G$ . We give a formula for this Poincaré series of a collection of plane valuations in terms of a  $G$ -resolution of the collection. We show that, for filtrations on the ring of germs of functions in two variables defined by the curve valuations corresponding to the irreducible components of a plane curve singularity defined by a  $G$ -invariant function germ, in the majority of cases this equivariant Poincaré series determines the corresponding equivariant monodromy zeta functions defined earlier.

## Introduction

It was shown (see, e.g., [1]) that the multi-variable Poincaré series of the multi-index filtration on the ring of germs of functions in two variables defined by the curve valuations corresponding to the irreducible components of a (reducible) plane curve singularity  $C$  coincides with the multi-variable Alexander polynomial of the corresponding algebraic link  $C \cap S_\varepsilon^3 \subset S_\varepsilon^3$ . Up to now this coincidence has no conceptual explanation. It is obtained by direct computations of the both objects in the same terms and comparison of the results. Generalizations of this relation (e.g., to an equivariant setting) can help to

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understand a reason for it. A possibility to try to get an equivariant generalization of this relation was restricted by the lack of equivariant versions of the Poincaré series of filtrations and of the monodromy zeta functions.

There were several attempts to define equivariant versions of the Poincaré series and of the monodromy zeta functions for a finite group  $G$  action: [2], [3], [6], [7]. In particular, in [2] an equivariant version of the Poincaré series was defined as an element of  $R_1(G)[[t_1, \dots, t_r]]$ , where  $R_1(G)$  is the subring of the ring of representations of the group  $G$  generated by the one-dimensional representations. This version appeared to be useful for computation of Seiberg-Witten invariants of the links of the so-called splice-quotient surface singularities (see, e.g., [10]). However a comparison of the equivariant versions of the Poincaré series and of the monodromy zeta functions was difficult since they were elements of different rings. The notion of the (usual, non-equivariant) Poincaré series is related to the notion of integration with respect to the Euler characteristic: see, e.g. [1]. An “equivariant version” of the ring  $\mathbb{Z}$  of integers is the Burnside ring  $A(G)$  of the group  $G$ , i.e. the Grothendieck ring of finite  $G$ -sets (see, e.g., [11]). In some approaches an equivariant version of the Euler characteristic is an element of the Burnside ring. In [6] and [7] equivariant versions of the monodromy zeta function were defined as elements of the power series rings  $(A(G) \otimes \mathbb{Q})[[t_1, \dots, t_r]]$  and  $A(G)[[t_1, \dots, t_r]]$  respectively.

Here we define a new equivariant version  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  of the Poincaré series as an element of  $\tilde{A}(G)[[t_1, \dots, t_r]]$  for a certain modification  $\tilde{A}(G)$  of the Burnside ring ( $\{\nu_i\}$  is a collection of order functions on the ring of germs of analytic functions of a complex analytic space). A reduction of this series is an element of  $A(G)[[t_1, \dots, t_r]]$ . We give a formula for the equivariant Poincaré series  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  of a collection  $\{\nu_i\}$  of plane valuations in terms of a  $G$ -resolution of the collection. This formula uses a pre- $\lambda$ -structure on the ring  $\tilde{A}(G)$ . We also correct a certain inaccuracy in the corresponding formula in [3] (and of its use in [4]). We show that (with some exceptions) the equivariant Poincaré series of a collection of curve valuations corresponding to the irreducible components of a plane curve singularity defined by a  $G$ -invariant equation determines the corresponding equivariant monodromy zeta-functions from [6] and [7] by a simple algorithm.

## 1 The equivariant Poincaré series

We shall consider the Grothendieck ring of finite  $G$ -sets with an additional structure.

**Definition:** A finite *equipped*  $G$ -set is a pair  $\tilde{X} = (X, \alpha)$  where:

- $X$  is a finite  $G$ -set;
- $\alpha$  associates to each point  $x \in X$  a one-dimensional representation  $\alpha_x$  of the isotropy subgroup  $G_x = \{a \in G : ax = x\}$  of the point  $x$  so that, for  $a \in G$ , one has  $\alpha_{ax}(b) = \alpha_x(a^{-1}ba)$ , where  $b \in G_{ax} = aG_xa^{-1}$ .

Let  $\tilde{A}(G)$  be the Grothendieck group of finite equipped  $G$ -sets (with respect to the disjoint union of equipped  $G$ -sets). The cartesian product  $\tilde{X} \times \tilde{Y}$  of two equipped  $G$ -sets  $\tilde{X} = (X, \alpha)$  and  $\tilde{Y} = (Y, \beta)$  is the pair  $(X \times Y, \gamma)$ , where  $X \times Y$  is the cartesian product of the  $G$ -sets  $X$  and  $Y$  with the diagonal  $G$ -action and  $\gamma_{(x,y)}(b) = \alpha_x(b)\beta_y(b)$  for  $b \in G_{(x,y)} = G_x \cap G_y$ . The cartesian product defines a ring structure on  $\tilde{A}(G)$ . The class of an equipped  $G$ -set  $\tilde{X}$  in the Grothendieck ring  $\tilde{A}(G)$  will be denoted by  $[\tilde{X}]$ . The unit 1 in the ring  $\tilde{A}(G)$  is represented by the one-point  $G$ -set  $G/G$  with the trivial representation of (the isotropy subgroup)  $G$  associated to it. As an Abelian group  $\tilde{A}(G)$  is freely generated by the classes of the irreducible equipped  $G$ -sets  $[G/H]_\alpha$  for all the conjugacy classes  $[H]$  of subgroups of  $G$  and for all one-dimensional representations  $\alpha$  of  $H$  (a representative of the conjugacy class  $[H] \in \text{Conjsub } G$ ).

**Example.** The Burnside ring  $A(\mathbb{Z}_2)$  of the cyclic group of order 2 is the free abelian group generated by the classes  $1 = [\mathbb{Z}_2/\mathbb{Z}_2]$  and  $[\mathbb{Z}_2/(e)]$ . The ring  $\tilde{A}(\mathbb{Z}_2)$  is the free abelian group generated the classes  $1$ ,  $[\mathbb{Z}_2/\mathbb{Z}_2]_\sigma$  and  $[\mathbb{Z}_2/(e)]$  where the class  $[\mathbb{Z}_2/\mathbb{Z}_2]_\sigma$  is represented by the one-point  $\mathbb{Z}_2$ -set  $\mathbb{Z}_2/\mathbb{Z}_2$  with the nontrivial representation of  $\mathbb{Z}_2$  associated to the point. The multiplication is defined by  $[\mathbb{Z}_2/\mathbb{Z}_2]_\sigma \cdot [\mathbb{Z}_2/(e)] = [\mathbb{Z}_2/(e)]$ .

Let  $R_1(G)$  be the subring of the ring  $R(G)$  of representations of the group  $G$  generated by the one-dimensional representations. (If the group  $G$  is Abelian one has  $R_1(G) = R(G)$ .) There are natural homomorphisms from the ring  $\tilde{A}(G)$  to the rings  $A(G)$ ,  $\mathbb{Z}$  and to  $R_1(G)$  respectively. The reduction  $\rho : \tilde{A}(G) \rightarrow A(G)$  is defined by forgetting the representation corresponding to the points. The reduction  $\hat{\rho} : \tilde{A}(G) \rightarrow \mathbb{Z}$  is defined by forgetting the representations and the  $G$ -action. (Thus one gets an element of the Grothendieck ring of finite sets isomorphic to  $\mathbb{Z}$ .) The homomorphism  $\varepsilon : \tilde{A}(G) \rightarrow R_1(G)$  is defined in the following way. For an equipped  $G$ -set  $\tilde{X} = (X, \alpha)$ , let  $X^G$  be the set of fixed points of the  $G$ -action. For each point  $x \in X^G$ ,  $\alpha_x$  is a one-dimensional representation of the group  $G$  (coinciding with the isotropy subgroup). Thus one gets a finite set with a one-dimensional representation of  $G$  associated to each point. The Grothendieck ring of such sets is  $R_1(G)$ . One define  $\varepsilon([\tilde{X}])$  as  $[(X^G, \alpha|_{X^G})] \in R_1(G)$ .

**Remark.** In [3] there were defined a notion of a (“locally finite”)  $(G, r)$ -set and the corresponding Grothendieck ring  $K_0((G, r) - \text{sets})$ . One can see that

a finite  $(G, r)$ -set is an equipped  $G$ -set with an additional structure (a  $\mathbb{Z}_{\geq 0}^r$ -valued function on it). Therefore each finite  $(G, r)$ -set defines a finite equipped  $G$ -set.

A pre- $\lambda$ -structure on a ring  $R$  is a map from  $R$  to  $1 + tR[[t]]$  ( $u \mapsto \lambda_u(t)$ ,  $u \in R$ ) such that  $\lambda_{u+v}(t) = \lambda_u(t)\lambda_v(t)$ : see e.g. [8]. In what follows we shall use a (particular) pre- $\lambda$ -structure on the ring  $\tilde{A}(G)$ . Since  $\lambda_1(t)$  will be equal to  $(1 - t)^{-1} = 1 + t + t^2 + \dots$ , we shall denote  $\lambda_u(t)$  by  $(1 - t)^{-u}$ . We shall define

$$\lambda_{[\tilde{X}]}(t) = (1 - t)^{-[\tilde{X}]} := 1 + [\tilde{X}]t + [S^2\tilde{X}]t^2 + \dots, \quad (1)$$

where  $S^k\tilde{X}$  are the symmetric powers of the equipped  $G$ -set  $\tilde{X}$  defined below.

For an equipped  $G$ -set  $\tilde{X} = (X, \alpha)$ , its symmetric power  $S^k\tilde{X}$  is an equipped  $G$ -set described in the following way. It is the pair  $(S^kX, \alpha^{(k)})$  where  $S^kX = X^k/S_k$  is the  $k$ -th symmetric power of the  $G$ -set  $X$  (with the natural action of  $G$  on it). Let an unordered collection  $\{x_1, \dots, x_k\}$  represent a point of  $S^kX$ . One can write it as  $\{\mu_1y_1, \dots, \mu_sy_s\}$  where  $y_1, \dots, y_s$  are different points from the collection  $x_1, \dots, x_k$  and  $\mu_i$  are their multiplicities,  $\sum \mu_i = k$ . The isotropy subgroup of the point  $\{x_1, \dots, x_k\} \in S^kX$  consists of those elements  $a \in G$  which act on the set  $y_1, \dots, y_s$  by a permutation preserving the multiplicities of the points. Let  $(y_{i_1}, \dots, y_{i_\ell})$  be a cycle of the permutation defined by the action of  $a$  on  $\{y_1, \dots, y_s\}$ . The multiplicities  $\mu_{i_1}, \dots, \mu_{i_\ell}$  are equal to each other. Let us define  $\beta(y_{i_1}, \dots, y_{i_\ell})$  as  $\alpha_{y_{i_1}}(a^\ell)$ . Now  $\alpha_{\{x_i\}}^{(k)}(a)$  is the product of the factors  $(\beta(y_{i_1}, \dots, y_{i_\ell}))^{\mu_{i_1}}$  over all the cycles  $(y_{i_1}, \dots, y_{i_\ell})$  in the permutation defined by  $a$ .

**Remark.** The reason for this definition is inspired by the application of this notion below and is the following one. One has to explain it for an irreducible equipped  $G$ -set: the  $k$ -th symmetric power of the disjoint union of two equipped  $G$ -sets  $\tilde{X}' = (X', \alpha')$  and  $\tilde{X}'' = (X'', \alpha'')$  is the disjoint union of the products  $S^i\tilde{X}' \times S^{k-i}\tilde{X}''$  over all  $i = 0, 1, \dots, k$ . Assume that the group  $G$  acts on a germ  $(V, 0)$  of a complex analytic space (and thus on the ring  $\mathcal{O}_{V,0}$  of germs of functions on it and on its projectivization  $\mathbb{P}\mathcal{O}_{V,0}$ ) and that an irreducible equipped  $G$ -set  $\tilde{X} = (X, \alpha)$  is represented by the orbit  $Gh$  of a function germ  $h \in \mathbb{P}\mathcal{O}_{V,0}$  (or rather of its class there). Let  $G_h$  be the isotropy subgroup of  $h$  in  $\mathbb{P}\mathcal{O}_{V,0}$ . This means that the  $G$ -set  $X = Gh \cong G/G_h$  consists of  $|G|/|G_h|$  points represented by a set of function germs, for an element  $a \in G$  and for a function  $h'$  from the set of function germs representing the orbit of  $h$ , the function  $a^*h'$  is, up to a constant factor, another function-germ from this set, and, for  $a \in G_{h'}$ ,  $a^*h' = \alpha_{h'}(a)h'$ . Let us consider all functions of the form  $h_1 \cdot h_2 \cdot \dots \cdot h_k$ , where  $h_1, h_2, \dots, h_k$  are functions from the set representing  $Gh$ . In the application below two products of this sort can coincide in  $\mathbb{P}\mathcal{O}_{V,0}$

if and only if they consist of the same functions (in an arbitrary order, of course). (This will be the case since all the functions from the orbit will have different zero-level curves.) In this case this set of functions is, in the natural way, isomorphic to the  $k$ -th symmetric power  $S^k X$  of the  $G$ -set  $X$ . Let us write a product of the functions representing  $Gh$  as  $\{h^* = h_1^{\mu_1} \cdot \dots \cdot h_s^{\mu_s}\}$  where  $h_1, \dots, h_s$  are different functions from the set representing  $Gh$  and  $\mu_i$  are their multiplicities. The isotropy subgroup of the function  $h^*$  in  $\mathbb{P}\mathcal{O}_{V,0}$  consists of those elements  $a \in G$  which act on the set  $h_1, \dots, h_s$  by a permutation preserving the multiplicities of the functions. The function  $h^*$  is the product of the functions corresponding to cycles of the permutation. Let  $(h_{i_1}, \dots, h_{i_\ell})$  be a cycle of the permutation defined by  $a$ . The multiplicities  $\mu_{i_1}, \dots, \mu_{i_\ell}$  of these functions in the product are equal to each other. The element  $a$  acts on the product  $\prod_{j=1}^{\ell} h_{i_j}$  by the multiplication by  $\alpha_{h_{i_1}}(a^\ell)$ . Thus the element  $a$  acts on the function  $h^*$  by the multiplication by the product of the factors  $(\alpha_{h_{i_1}}(a^\ell))^{\mu_{i_1}}$  over all the cycles  $(y_{i_1}, \dots, y_{i_\ell})$  in the permutation defined by  $a$ .

From the definition (1) one can see that

$$(1-t)^{-[\tilde{X} \cup \tilde{Y}]} = (1-t)^{-[\tilde{X}]}(1-t)^{-[\tilde{Y}]}.$$

Therefore this definition in the obvious way extends to the Grothendieck ring  $\tilde{A}(G)$  defining a pre- $\lambda$ -structure on it.

**Example.** For  $G = \mathbb{Z}_2$ ,

$$\begin{aligned} (1-t)^{[\mathbb{Z}_2/\mathbb{Z}_2]_\sigma} &= 1 + [\mathbb{Z}_2/\mathbb{Z}_2]_\sigma t + t^2 + [\mathbb{Z}_2/\mathbb{Z}_2]_\sigma t^3 + \dots = \frac{1 + [\mathbb{Z}_2/\mathbb{Z}_2]_\sigma t}{1-t^2}, \\ (1-t)^{[\mathbb{Z}_2/(e)]} &= 1 + [\mathbb{Z}_2/(e)]t + ([\mathbb{Z}_2/(e)] + 1)t^2 + \dots \\ &\quad \dots + (k[\mathbb{Z}_2/(e)] + 1)t^{2k} + (k+1)[\mathbb{Z}_2/(e)]t^{2k+1} + \dots \\ &= \frac{1}{1-t^2} + \frac{t}{(1-t)(1-t^2)}[\mathbb{Z}_2/(e)]. \end{aligned}$$

Let  $(V, 0)$  be a germ of a complex analytic space with an action of a finite group  $G$  and let  $\mathcal{O}_{V,0}$  be the ring of germs of functions on it. Without loss of generality we assume that the  $G$ -action on  $(V, 0)$  is faithful. The group  $G$  acts on  $\mathcal{O}_{V,0}$  by

$$a^* f(z) = f(a^{-1}z)$$

where  $z \in V$  and  $a \in G$ .

A valuation  $\nu$  on the ring  $\mathcal{O}_{V,0}$  is a function  $\nu : \mathcal{O}_{V,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$  such that:

- 1)  $\nu(\lambda f) = \nu(f)$  for  $\lambda \in \mathbb{C}^*$ ;
- 2)  $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ ;
- 3)  $\nu(fg) = \nu(f) + \nu(g)$ .

A function  $\nu : \mathcal{O}_{V,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$  which possesses the properties 1) and 2) is called an *order function*. A collection of order functions  $\nu_1, \dots, \nu_r$  on  $\mathcal{O}_{V,0}$  defines an  $r$ -index filtration on  $\mathcal{O}_{V,0}$ :

$$J(\underline{v}) = \{h \in \mathcal{O}_{V,0} : \underline{\nu}(h) \geq \underline{v}\},$$

where  $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}_{\geq 0}^r$ ,  $\underline{\nu}(h) = (\nu_1(h), \dots, \nu_r(h))$  and  $\underline{v}' = (v'_1, \dots, v'_r) \geq \underline{v}'' = (v''_1, \dots, v''_r)$  if and only if  $v'_i \geq v''_i$  for all  $i$ .

The notion of the Poincaré series  $P_{\{\nu_i\}}(\underline{t})$  of the filtration  $\{J(\underline{v})\}$  (or of the collection  $\{\nu_i\}$  of order functions) was introduced in [5]. In [1] it was shown that it is equal to a certain integral with respect to the Euler characteristic over the projectivization  $\mathbb{P}\mathcal{O}_{V,0}$  of the space  $\mathcal{O}_{V,0}$ . The coefficient at  $\underline{t}^{\underline{v}}$  ( $\underline{t} = (t_1, \dots, t_r)$ ,  $\underline{v} = (v_1, \dots, v_r)$ ,  $\underline{t}^{\underline{v}} = t_1^{v_1} \cdot \dots \cdot t_r^{v_r}$ ) in the Poincaré series  $P_{\{\nu_i\}}(\underline{t})$  is equal to the Euler characteristic of the set of function germs  $h \in \mathbb{P}\mathcal{O}_{V,0}$  such that  $\underline{\nu}(h) = \underline{v}$ .

One of the problems to define an equivariant version of the Poincaré series is related to the fact that an order function  $\nu$  on  $\mathcal{O}_{V,0}$  is, in general, not invariant with respect to the  $G$ -action. Therefore a  $G$ -orbit of a function does not correspond to a well-defined monomial of the form  $\underline{t}^{\underline{v}}$ . One can restrict oneself to only  $G$ -equivariant order functions on  $\mathcal{O}_{V,0}$ . However this makes the construction rather poor. Instead of that one can associate to an orbit the sum of values of the  $G$ -shifts of the order function, i.e. the sum of the values of the order functions  $a^*\nu$  for  $a \in G$ . This leads to a meaningful notion. E.g., if  $\nu_1, \dots, \nu_r$  are the curve valuations defined by irreducible plane curve singularities  $(C_i, 0) \subset (\mathbb{C}^2, 0)$  (see Section 2) and  $\mu(h) := \nu_1(h) + \dots + \nu_r(h)$  then the integral  $\int_{\mathbb{P}\mathcal{O}_{V,0}} t^{\mu(h)} d\chi$  is equal to  $\Delta^C(t, \dots, t)$ , where  $\Delta^C$  is the Alexander polynomial (in several variables) of the link of the curve singularity  $C = \bigcup_{i=1}^r C_i$ , which in its turn coincides with the monodromy zeta function of an equation of the curve  $C$ .

The group  $G$  acts on the set of (order) functions on  $\mathcal{O}_{V,0}$ . Let  $\nu_1, \dots, \nu_r$  be a collection of order functions on  $\mathcal{O}_{V,0}$ . Let  $\omega_i : \mathcal{O}_{V,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$  be defined by  $\omega_i = \sum_{a \in G} a^*\nu_i$ . The functions  $\omega_i$  are  $G$ -invariant.

The functions  $\omega_i$  are not, in general, order functions. Assume that the order functions  $\nu_1, \dots, \nu_r$  are finitely determined. This means that, for each  $k \in \mathbb{Z}_{\geq 0}$ , the set  $\{h \in \mathcal{O}_{V,0} : \nu_i(h) = k\}$  is cylindric in the sense of [1]. For an element  $h \in \mathbb{P}\mathcal{O}_{V,0}$ , that is for a function germ considered up to a constant factor, let  $G_h$  be the isotropy subgroup  $G_h = \{a \in G : a^*h = \alpha_h(a)h\}$

( $a \mapsto \alpha_h(a) \in \mathbb{C}^*$  determines a one-dimensional representation of the subgroup  $G_h$ ) and let  $Gh \cong G/G_h$  be the orbit of  $h$  in  $\mathbb{P}\mathcal{O}_{V,0}$ . Let  $\tilde{X}_h$  be the element of the ring  $\tilde{A}(G)$  represented by the  $G$ -set  $Gh$  with the representation  $\alpha_{a^*h}$  associated to the point  $a^*h \in Gh$  ( $a \in G$ ). The correspondence  $h \mapsto \tilde{X}_h$  defines a function ( $\tilde{X}$ ) on  $\mathbb{P}\mathcal{O}_{V,0}/G$  with values in  $\tilde{A}(G)$ .

The usual (non-equivariant) Poincaré series of a collection of order functions  $\{\nu_i\}$  is the integral with respect to the Euler characteristic over the projectivization  $\mathbb{P}\mathcal{O}_{V,0}$  of the function  $\underline{t}^{\underline{\nu}(h)}$  with values in  $\mathbb{Z}[[t_1, \dots, t_r]]$ . The equivariant version will be defined as the integral over  $\mathbb{P}\mathcal{O}_{V,0}/G$  of the function  $\tilde{X}_h \underline{t}^{\underline{\omega}(h)}$  with values in  $\tilde{A}(G)[[t_1, \dots, t_r]]$ : see below. However one has to make the notion of integration with respect to the Euler characteristic over  $\mathbb{P}\mathcal{O}_{V,0}/G$  more precise. The reason is that the function  $\tilde{X}_h \underline{t}^{\underline{\omega}(h)}$  (or rather  $\tilde{X}_h$  itself) is not cylindric: the condition that  $G_h = H$  is not determined by a jet of the germ  $h$  of any order. Therefore one has to change the notion of measurable subsets of  $\mathbb{P}\mathcal{O}_{V,0}/G$  (i.e., of those subsets for which the Euler characteristic is defined) a little bit.

The quotient  $\mathbb{P}\mathcal{O}_{V,0}/G$  is decomposed into the disjoint parts  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}$  for all the conjugacy classes  $[H]$  of subgroups of  $G$  and all the one-dimensional representations  $\alpha$  of  $H$ , where  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}$  consists of the functions  $h \in \mathbb{P}\mathcal{O}_{V,0}/G$  with the isotropy subgroup  $G_h$  conjugate to  $H$  and, for those of them with  $G_h = H$ , the corresponding one-dimensional representation of  $H$  is  $\alpha$ . Let  $\overline{(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}}$  be the set of  $h \in \mathbb{P}\mathcal{O}_{V,0}/G$  such that  $H$  is conjugate to a subgroup of  $G_h$  and, for those of them with  $H \subset G_h$ , the restriction of the corresponding one-dimensional representation of  $G_h$  to  $H$  coincides with  $\alpha$ . (The set  $\overline{(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}}$  is in some sense the closure of  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}$ .) In the usual way one can define measurable (i.e., cylindric) subsets of  $\overline{(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}}$ . Now a subset  $A$  of  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}$  will be called measurable if it is the intersection with  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}$  of a cylindric subset  $B$  in  $\mathbb{P}\mathcal{O}_{V,0}/G$  ( $= \overline{(\mathbb{P}\mathcal{O}_{V,0}/G)^{[(e)],1}}$ ). Its measure (the Euler characteristic) is defined by the following recurrent equation

$$\chi(A) = \chi\left(B \cap \overline{(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H],\alpha}}\right) - \sum \chi\left(B \cap (\mathbb{P}\mathcal{O}_{V,0}/G)^{[H'],\alpha'}\right),$$

where the last sum is over all the conjugacy classes  $[H'] \in \text{Conjsub } G$  such that  $H \subset H'$ ,  $H \neq H'$  and  $\alpha'_H = \alpha$ . (This equation is a recurrent one since it assumes that the measures (the Euler characteristics) of subsets of  $(\mathbb{P}\mathcal{O}_{V,0}/G)^{[H'],\alpha'}$  with  $H' \supset H$ ,  $H' \neq H$  are already defined.

**Definition:** The *equivariant Poincaré series*  $P_{\{\nu_i\}}^G(\underline{t})$  of the collection  $\{\nu_i\}$  is



defined by

$$P_{\{\nu_i\}}^G(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}/G} \tilde{X}_h \underline{t}^{\underline{\omega}(h)} d\chi \in \tilde{A}(G)[[t_1, \dots, t_r]], \quad (2)$$

where  $\underline{t}^{\underline{\omega}(h)} = t_1^{\omega_1(h)} \cdot \dots \cdot t_r^{\omega_r(h)}$ ,  $t_i^{+\infty}$  should be regarded as 0.

In other words

$$P_{\{\nu_i\}}^G(\underline{t}) = \sum \chi(\{h \in \mathbb{P}\mathcal{O}_{V,0} : \underline{\omega}(h) = \underline{w}, \tilde{X}_h = [G/H]_\alpha\}/G)[G/H]_\alpha \underline{t}^{\underline{w}},$$

where the sum is over all the conjugacy classes  $[H]$  of subgroups in  $G$ , all the one-dimensional representations  $\alpha$  of  $H$  and  $\underline{w} \in \mathbb{Z}_{\geq 0}^r$ .

Applying the reduction homomorphism  $\rho : \tilde{A}(G) \rightarrow A(G)$  to the Poincaré series  $P_{\{\nu_i\}}^G(\underline{t})$ , i.e. to its coefficients, one gets the series  $\rho P_{\{\nu_i\}}^G(\underline{t}) \in A(G)[[t_1, \dots, t_r]]$ , i.e. a power series with the coefficients from the (usual) Burnside ring. Applying the homomorphism  $\hat{\rho} : \tilde{A}(G) \rightarrow \mathbb{Z}$  and  $\varepsilon : \tilde{A}(G) \rightarrow R_1(G)$ , one gets the series  $\hat{\rho} P_{\{\nu_i\}}^G(\underline{t}) \in \mathbb{Z}[[t_1, \dots, t_r]]$  and  $\varepsilon P_{\{\nu_i\}}^G(\underline{t}) \in R_1(G)[[t_1, \dots, t_r]]$  respectively.

**Statement 1** *One has*

$$\hat{\rho} P_{\{\nu_i\}}^G(\underline{t}) = P_{\{a * \nu_i\}}(t_1, \dots, t_1, t_2, \dots, t_2, \dots, t_r, \dots, t_r),$$

where  $P_{\{a * \nu_i\}}(\cdot)$  is the usual (non-equivariant) Poincaré series of the collection of  $|G|r$  order functions  $\{a * \nu_1, a * \nu_2, \dots, a * \nu_r | a \in G\}$  (each group of equal variables in  $P_{\{a * \nu_i\}}$  consists of  $|G|$  of them). One has

$$\varepsilon P_{\{\nu_i\}}^G(\underline{t}) = P^G(t_1^{|G|}, t_2^{|G|}, \dots, t_r^{|G|}),$$

where  $P^G(\underline{t})$  is the equivariant Poincaré series defined in [2] (actually only for divisorial and curve valuations on  $\mathcal{O}_{\mathbb{C}^2,0}$ ) as an element of the power series ring  $R_1(G)[[t_1, t_2, \dots, t_r]]$ .

In [3] an equivariant version of the Poincaré series was defined not as a power series, but as an element of a rather big (and sophisticated) Grothendieck ring of so-called locally finite  $(G, r)$ -sets ( $G$ -sets with an additional structure).

A *locally finite*  $(G, r)$ -set is a triple  $(X, \underline{v}, \alpha)$  where

- $X$  is a  $G$ -set;
- $\underline{v}$  is a function on  $X$  with values in  $\mathbb{Z}_{\geq 0}^r$ ;
- $\alpha$  associates to each point  $x \in X$  a one-dimensional representation  $\alpha_x$  of the isotropy subgroup  $G_x = \{a \in G : ax = x\}$  of the point  $x$ ;



satisfying the following conditions:

- 1)  $\alpha_{ax}(b) = \alpha_x(a^{-1}ba)$  for  $x \in X$ ,  $a \in G$ ,  $b \in G_{ax} = aG_xa^{-1}$ ;
- 2) for any  $\underline{v} \in \mathbb{Z}_{\geq 0}^r$  the set  $\{x \in X : \underline{v}(x) \leq \underline{v}\}$  is finite.

The equivariant version of the Poincaré series in [3] is a virtual locally finite  $(G, r)$ -set  $P_{\{\nu_i\}}^G = (X, \underline{v}, \alpha)$ . Consider the function  $\underline{\omega}$  on  $X$  with values in  $\mathbb{Z}_{\geq 0}^r$  defined by

$$\underline{\omega}(x) = \sum_{a \in G} \underline{v}(ax).$$

This function is  $G$ -invariant and thus is constant on irreducible components of the  $(G, r)$ -set. For  $\underline{w} \in \mathbb{Z}_{\geq 0}^r$ , let  $X_{\underline{w}} = \{x \in X : \underline{\omega}(x) = \underline{w}\}$ . The  $G$ -set  $X_{\underline{w}}$  with the representations  $\alpha_x$  associated to its points is a finite equipped  $G$ -set  $\tilde{X}_{\underline{w}} = (X_{\underline{w}}, \alpha|_{X_{\underline{w}}})$ . One can see that the equivariant Poincaré series (2) is equal to

$$P_{\{\nu_i\}}^G(\underline{t}) = \sum_{\underline{w} \in \mathbb{Z}_{\geq 0}^r} [(X_{\underline{w}}, \alpha|_{X_{\underline{w}}})] \underline{t}^{\underline{w}}.$$

Thus the equivariant version of the Poincaré series from [3] determines the Poincaré series considered here.

## 2 The equivariant Poincaré series for curve and divisorial valuations on $\mathcal{O}_{\mathbb{C}^2, 0}$

Here we write an A'Campo type formula for the equivariant Poincaré series mentioned in the title of the section in terms of a  $G$ -resolution. We shall treat two types of plane valuations.

Let  $(C, 0) \subset (\mathbb{C}^2, 0)$  be an irreducible germ of a plane curve and let  $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$  be a parametrization (uniformization) of it (i.e.  $\text{Im } \varphi = (C, 0)$  and  $\varphi$  is an isomorphism between  $(\mathbb{C}, 0)$  and  $(C, 0)$  outside of the origin). For  $h \in \mathcal{O}_{\mathbb{C}^2, 0}$ , let  $h(\varphi(\tau)) = c\tau^{\nu(h)} + \text{terms of higher degree}$ , where  $c \neq 0$ ,  $\nu(h) \in \mathbb{Z}_{\geq 0}$ . (If  $h(\varphi(\tau)) \equiv 0$ , one defines  $\nu(h)$  as  $+\infty$ .) The function  $\nu$  on  $\mathcal{O}_{\mathbb{C}^2, 0}$  is a valuation: the so-called curve valuation.

Let  $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  be a modification of the plane  $(\mathbb{C}^2, 0)$  by a sequence of blowing-ups. Its exceptional divisor  $\mathcal{D} = \pi^{-1}(0)$  is the union of irreducible components  $E_\sigma$ ,  $\sigma \in \Gamma$ , each of them is isomorphic to the complex projective line  $\mathbb{CP}^1$ . For a component  $E_\sigma$  of the exceptional divisor, and for  $h \in \mathcal{O}_{\mathbb{C}^2, 0}$ , let  $\nu_\sigma(h)$  be the order of zero of the lifting  $h \circ \pi$  of the function  $h$  to the space  $\mathcal{X}$  of the modification along the component  $E_\sigma$ . The function  $\nu_\sigma$

on  $\mathcal{O}_{\mathbb{C}^2,0}$  is a valuation: the so-called divisorial valuation (corresponding to the divisor  $E_\sigma$ ).

Assume that a finite group  $G$  acts on  $(\mathbb{C}^2, 0)$  (by a representation). Let  $\nu_i$ ,  $i = 1, \dots, r$ , be either a curve or a divisorial valuation on  $\mathcal{O}_{\mathbb{C}^2,0}$ . We shall write  $I_0 = \{1, 2, \dots, r\} = I' \sqcup I''$ , where  $i \in I'$  if and only if the corresponding valuation  $\nu_i$  is a curve one. For  $i \in I'$ , let  $(C_i, 0)$  be the plane curve defining the valuation  $\nu_i$ .

A *G-equivariant resolution* (or a *G-resolution* for short) of the collection  $\{\nu_i\}$  of valuations is a proper complex analytic map  $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  from a smooth surface  $\mathcal{X}$  with a  $G$ -action such that:

- 1)  $\pi$  is an isomorphism outside of the origin in  $\mathbb{C}^2$ ;
- 2)  $\pi$  commutes with the  $G$ -actions on  $\mathcal{X}$  and on  $\mathbb{C}^2$ ;
- 3) the total transform  $\pi^{-1}(\bigcup_{i \in I', a \in G} aC_i)$  of the curve  $GC = G(\bigcup_{i \in I'} C_i)$  is a normal crossing divisor on  $\mathcal{X}$  (in particular, the exceptional divisor  $\mathcal{D} = \pi^{-1}(0)$  is a normal crossing divisor as well);
- 4) for each branch  $C_i$ ,  $i \in I'$ , its strict transform  $\tilde{C}_i$  is a germ of a smooth curve transversal to the exceptional divisor  $\mathcal{D}$  at a smooth point  $x$  of it and is invariant with respect to the isotropy subgroup  $G_x = \{g \in G : gx = x\}$  of the point  $x$ ;
- 5) for each  $i \in I''$ , the exceptional divisor  $\mathcal{D} = \pi^{-1}(0)$  contains the divisor defining the divisorial valuation  $\nu_i$ .

A  $G$ -resolution can be obtained by a  $G$ -invariant sequence of blow-ups of points. The condition 4) means, in particular, that  $\pi$  is an embedded resolution of the curve  $GC = \bigcup_{i \in I', a \in G} aC_i$ .

Let  $\mathring{\mathcal{D}}$  be the “smooth part” of the exceptional divisor  $\mathcal{D}$  in the total transform  $\pi^{-1}(GC)$  of the curve  $GC$ , i.e.,  $\mathcal{D}$  itself minus all the intersection points of its components and all the intersection points with the components of the strict transform of the curve  $GC$ . For  $x \in \mathring{\mathcal{D}}$ , let  $\tilde{L}_x$  be a germ of a smooth curve on  $\mathcal{X}$  transversal to  $\mathring{\mathcal{D}}$  at the point  $x$  and invariant with respect to the isotropy subgroup  $G_x$  of the point  $x$ . The image  $L_x = \pi(\tilde{L}_x) \subset (\mathbb{C}^2, 0)$  is called a *curvette* at the point  $x$ . Let the curvette  $L_x$  be given by an equation  $h_x = 0$ ,  $h_x \in \mathcal{O}_{\mathbb{C}^2,0}$ . Without loss of generality one can assume that the function germ  $h_x$  is  $G$ -equivariant. Moreover we shall assume that the germs  $h_x$  associated to different points  $x \in \mathring{\mathcal{D}}$  are chosen so that  $h_{ax}(a^{-1}z)/h_x(z)$  is a constant (depending on  $a$  and  $x$ ). Also we shall assume that the dependence of the germ

$h_x$  on the point  $x$  is constructible, i.e. depends on  $x$  analytically on  $x \in \mathring{\mathcal{D}}$  except at a finite set of points.

Let  $E_\sigma$ ,  $\sigma \in \Gamma$ , be the set of all irreducible components of the exceptional divisor  $\mathcal{D}$  ( $\Gamma$  is a  $G$ -set itself). For  $\sigma$  and  $\delta$  from  $\Gamma$ , let  $m_{\sigma\delta} := \nu_\sigma(h_x)$ , where  $\nu_\sigma$  is the corresponding divisorial valuation,  $h_x$  is the germ defining the curvette at a point  $x \in E_\delta \cap \mathring{\mathcal{D}}$ . One can show that the matrix  $(m_{\sigma\delta})$  is minus the inverse matrix to the intersection matrix  $(E_\sigma \circ E_\delta)$  of the irreducible components of the exceptional divisor  $\mathcal{D}$ . For  $i = 1, \dots, r$ , let  $m_{\sigma i} := m_{\sigma\delta}$ , where  $\delta$  is the number of the component of  $\mathcal{D}$  corresponding to the valuation  $\nu_i$ , i.e. either the component defining the valuation  $\nu_i$  if  $\nu_i$  is a divisorial valuation (i.e. if  $i \in I''$ ), or the component intersecting the strict transform of the corresponding irreducible curve  $C_i$  if  $\nu_i$  is a curve valuation (i.e. if  $i \in I'$ ). Let  $\underline{m}_\sigma := (m_{\sigma 1}, \dots, m_{\sigma r}) \in \mathbb{Z}_{\geq 0}^r$ .

Let  $\{\Xi\}$  be a stratification of the smooth curve  $\widehat{\mathcal{D}} = \mathring{\mathcal{D}}/G$  such that:

- 1) each stratum  $\Xi$  is connected;
- 2) for each point  $\widehat{x} \in \Xi$  and for each point  $x$  from its pre-image  $p^{-1}(\widehat{x})$ , the conjugacy class of the isotropy subgroup  $G_x$  of the point  $x$  is the same, i.e., depends only on the stratum  $\Xi$ .

The latter is equivalent to say that the factorization map  $p : \mathring{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$  is a covering over each stratum  $\Xi$ .

For a point  $x \in \mathring{\mathcal{D}}$ , let  $\widetilde{X}_x$  be the equipped  $G$ -set defined by  $\widetilde{X}_x = \widetilde{X}_{h_x}$ , where  $h_x$  is the  $G_x$ -equivariant function defining the choosen curvette at the point  $x$  (see above). The equipped  $G$ -set  $\widetilde{X}_x$  is one and the same for all points  $x$  from the preimage of a stratum  $\Xi$  and therefore it defines an element  $[\widetilde{X}_\Xi] \in \widetilde{A}(G)$ . Let

$$\underline{\omega}_x := \underline{\nu} \left( \prod_{a \in G} h_{ax} \right). \quad (3)$$

One can see that, for a stratum  $\Xi$  of the stratification,  $\underline{\omega}_x$  is one and the same for all points  $x$  from the preimage  $\pi^{-1}(\Xi)$ . We shall denote it by  $\underline{\omega}_\Xi$ .

**Theorem 1** *For a collection  $\{\nu_i\}$  of curve and divisorial valuations one has*

$$P_{\{\nu_i\}}^G(\underline{t}) = \prod_{\Xi} (1 - \underline{t}^{\underline{\omega}_\Xi})^{-\chi(\Xi)[\widetilde{X}_\Xi]}. \quad (4)$$

**Proof.** Let  $Y$  be the configuration space of effective divisors on  $\mathring{\mathcal{D}}$ . The space  $Y$  can be represented as

$$Y = \bigsqcup_{\{k_\alpha\}} \left( \prod_{\sigma \in \Gamma} S^{k_\alpha} \overset{\circ}{E}_\sigma \right) = \prod_{\sigma \in \Gamma} \left( \bigsqcup_{k=0}^{\infty} S^k \overset{\circ}{E}_\sigma \right),$$

where  $\overset{\circ}{E}_\sigma = E_\sigma \cap \overset{\circ}{\mathcal{D}}$ ,  $S^k Z = Z^k / S^k$  is the  $k$ -th symmetric power of the space  $Z$ . One has the natural  $G$ -action on the space  $Y$ . Let  $\hat{Y} = Y/G$  be the space of  $G$ -orbits in  $Y$ .

For a point  $y \in Y$ ,  $y = \sum_{i=1}^n x_i$ , let  $\tilde{X}_y$  be the element of the ring  $\tilde{A}(G)$  defined by  $\tilde{X}_y = \tilde{X}_{\prod_i h_{x_i}}$ . This way one gets a  $G$ -invariant map  $\tilde{X} : Y \rightarrow \tilde{A}(G)$  and thus a map  $\hat{X} : \hat{Y} \rightarrow \tilde{A}(G)$ .

Let  $\bar{Y}$  be the configuration space of effective divisors on the smooth curve  $\hat{\mathcal{D}} = \overset{\circ}{\mathcal{D}} / G$ . The space  $\bar{Y}$  can be represented as

$$\bar{Y} = \bigsqcup_{\{k_\Xi\}} \left( \prod_{\Xi} S^{k_\Xi} \Xi \right) = \prod_{\Xi} \left( \bigsqcup_{k=0}^{\infty} S^k \Xi \right).$$

Let  $\bar{X} : \bar{Y} \rightarrow \tilde{A}(G)$  be the function on  $\bar{Y}$  defined in the following way. Let  $\bar{y} = \sum_{i=1}^s \ell_i \hat{x}_i$ , where  $\hat{x}_i$  are different points from  $\hat{\mathcal{D}} = \overset{\circ}{\mathcal{D}} / G$ . Then

$$\bar{X}(\bar{y}) := \prod_{i=1}^s [S^{\ell_i} \tilde{X}_{h_{x_i}}],$$

where  $x_i$  is a point from the preimage  $p^{-1}(\hat{x}_i)$  of  $\hat{x}_i$ .

There is a natural map  $q : Y \rightarrow \bar{Y}$  which sends a point  $y = \sum_{i=1}^n x_i \in Y$  to the point  $\bar{y} = \sum_{i=1}^n \hat{x}_i$ , where  $\hat{x}_i = p(x_i)$  is the orbit of the point  $x_i$ . Let  $\underline{\omega}_{\bar{y}} := \sum_{i=1}^n \underline{\omega}_{x_i}$ , where  $\underline{\omega}_x$  is  $\underline{\omega}(h_x)$  (note that  $\underline{\omega}_x$  depends only on the component of  $\overset{\circ}{\mathcal{D}}$  containing  $x$ ).

The preimage of a point  $\bar{y} \in \bar{Y}$  in  $Y$  can be described in the following way. Let  $\bar{y} = \sum_{i=1}^s \ell_i \hat{x}_i$ , where  $\hat{x}_i$  are different points from  $\hat{\mathcal{D}} = \overset{\circ}{\mathcal{D}} / G$ . Then  $q^{-1}(\bar{y}) = \prod_{i=1}^s S^{\ell_i}(p^{-1}(\hat{x}_i))$ . The definition of the one-dimensional representation associated to points of the symmetric powers of an equipped  $G$ -set (or rather the explanation of its meaning after the definition) shows that as an equipped  $G$ -set  $q^{-1}(\bar{y})$  is isomorphic to  $\bar{X}(\bar{y})$

Thus one has:

$$\begin{aligned} \int_{\bar{Y}} \bar{X}(\bar{y}) \underline{t}^{\underline{\omega}_{\bar{y}}} d\chi &= \prod_{\Xi} \left\{ \sum_{k=0}^{\infty} \left( \sum_{\{k_i\} : \sum i k_i = k} \chi \left( \frac{\Xi \sum k_i \setminus \Delta}{\prod_i S_{k_i}} \right) \prod_i [S^i X_{\Xi}]^{k_i} \right) \underline{t}^{k \omega_{\Xi}} \right\} \\ &= \prod_{\Xi} \left\{ \sum_{k=0}^{\infty} \left( \sum_{\{k_i\}} \frac{\chi(\Xi)(\chi(\Xi) - 1) \cdots (\chi(\Xi) - \sum k_i + 1)}{\prod_i (k_i!)} \prod_i [S^i X_{\Xi}]^{k_i} \right) \underline{t}^{k \omega_{\Xi}} \right\} \\ &= \prod_{\Xi} (1 + [X_{\Xi}] \underline{t}^{\omega_{\Xi}} + [S^2 X_{\Xi}] \underline{t}^{2 \omega_{\Xi}} + \cdots)^{\chi(\Xi)}, \end{aligned}$$

where  $\Delta$  is the big diagonal in  $\Xi^{\sum k_i}$ , i.e. the set of points  $(x_1, \dots, x_{\sum k_i}) \in \Xi^{\sum k_i}$  with at least two coinciding components  $x_j$ . The last equation follows from the well-known equation

$$1 + \sum_{k=0}^{\infty} \left( \sum_{\{k_i\}: \sum i k_i = k} \frac{m(m-1)(m-2) \cdots (m - \sum k_i + 1)}{\prod_i (k_i!)} \prod_i a_i^{k_i} \right) t^k = (1 + a_1 t + a_2 t^2 \cdots)^m,$$

see, e.g., [9].

According to the described pre- $\lambda$ -structure on the ring  $\tilde{A}(G)$  one has:

$$\int_{\overline{Y}} \overline{X}(\overline{y}) \underline{t}^{\underline{\omega}_{\overline{y}}} d\chi = \prod_{\Xi} (1 - \underline{t}^{\underline{\omega}_{\Xi}})^{-\chi(\Xi)[X_{\Xi}]}.$$

It is sufficient to prove the equation (4) modulo the terms of degree greater than  $\underline{W}$  for an arbitrary  $\underline{W} \in \mathbb{Z}_{\geq 0}^r$ . We can assume that the resolution  $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  is such that for any  $h \in \mathcal{O}_{\mathbb{C}^2, 0}$  with  $\underline{w}(h) \leq \underline{W}$ , the strict transform of the curve  $\{h = 0\}$  intersects the exceptional divisor  $\mathcal{D}$  only at points of  $\mathring{\mathcal{D}}$ . Such resolution can be obtained from an arbitrary one by additional  $G$ -invariant blowing-ups of intersection points of components of the total transform  $\pi^{-1}(GC)$  of the curve  $GC$ . These blowing-ups add, to the stratification  $\{\Xi\}$  of  $\widehat{\mathcal{D}}$ , strata with zero Euler characteristic and thus do not change the right hand side of (4). Let  $\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}} = \{h \in \mathcal{O}_{\mathbb{C}^2, 0} : \underline{w}(h) \leq \underline{W}\}$ . Let  $I : \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}} \rightarrow Y$  be the map which sends a function  $h \in \mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}}$  to the set of intersection points of the strict transform of the zero-level curve  $\{h = 0\}$  with the exceptional divisor  $\mathcal{D}$  counted with the corresponding multiplicities. One has the commutative diagram:

$$\begin{array}{ccc} \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}} & \xrightarrow{I} & Y \\ p \downarrow & & \downarrow p \\ \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}}/G & \xrightarrow{\widehat{I}} & \widehat{Y} \end{array}$$

The preimages  $I^{-1}(y)$  of points from  $Y$  are affine subspaces of  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}}$  (see Proposition 2 in [1]) and thus have Euler characteristic equal to 1. This implies that the Euler characteristics of the preimages with respect to  $\widehat{I}$  of points of  $\widehat{Y}$  in  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}}/G$  are also equal to 1.

One has  $\underline{\omega} = \widehat{\omega} \circ \widehat{I}$ , but, in general,  $\widetilde{X} \neq \widehat{X} \circ \widehat{I}$  because the isotropy subgroup of  $h \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{W}}$  can be different from the isotropy subgroup of its image in  $Y$

(being a proper subgroup of the latter). Therefore the  $G$ -orbits of a point in  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}^W$  and of its image in  $Y$  can be different as  $G$ -sets.

To compute the integral of the function  $\tilde{X}_h t^{\omega(h)}$  over  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}^W/G$ , we shall, for each point  $\hat{y} \in \hat{Y}$ , construct a point  $h_{\hat{y}}$  in  $\hat{I}^{-1}(\hat{y})$  so that  $\tilde{X}_{h_{\hat{y}}} = \tilde{X}(\tilde{y}) = \hat{X}(\hat{y})$  and the complement  $\hat{I}^{-1}(\hat{y}) \setminus \{h_{\hat{y}}\}$  is fibred into  $\mathbb{C}^*$  families of points with the function  $\tilde{X}_h$  constant along the fibres. This implies that

$$\int_{\hat{I}^{-1}(h)} \tilde{X}_h d\chi = \hat{X}(\hat{y})$$

and the Fubini formula applied to the map  $\hat{I}$  gives (up to terms of degree greater than  $\underline{W}$ )

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}/G} \tilde{X}_h t^{\omega(h)} d\chi = \int_{\hat{Y}} \hat{X}(\hat{y}) t^{\omega_{\hat{y}}} d\chi = \int_{\bar{Y}} \bar{X}(\bar{y}) t^{\omega_{\bar{y}}} d\chi = \prod_{\Xi} (1 - t^{\omega_{\Xi}})^{-\chi(\Xi)[X_{\Xi}]}.$$

Let  $\hat{y} \in \hat{Y}$  be the orbit of  $y = \sum_{i=1}^m x_i \in Y$  and let  $h_y := \prod_{i=1}^m h_{x_i}$ . The isotropy subgroup of  $h_y$  in  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$  coincides with the isotropy subgroup of  $y$  in  $Y$  and therefore  $\hat{X}_{h_{\hat{y}}} = \hat{X}(\hat{y})$  (here  $h_{\hat{y}} = p(h_y)$  is the image of  $h_y$  in  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}/G$ ).

Let  $h \in I^{-1}(y)$ . The strict transforms of the curves  $\{h = 0\}$  and  $\{h_y = 0\}$  intersect the exceptional divisor  $\mathcal{D}$  at the same points with the same multiplicities. Therefore the ratio  $\psi = \frac{h \circ \pi}{h_y \circ \pi}$  of the liftings  $h \circ \pi$  and  $h_y \circ \pi$  of the functions  $h$  and  $h_y$  to the space  $\mathcal{X}$  of the modification has neither zeros nor poles on the exceptional divisor  $\mathcal{D}$  and thus is constant on it. Therefore (multiplying  $h$  by a constant) one can assume that the ratio  $\psi$  is equal to 1 on  $\mathcal{D}$ .

Let  $h_{\lambda} := h_y + \lambda(h - h_y)$ ,  $\lambda \in \mathbb{C}^*$ . One has  $\frac{h_{\lambda} \circ \pi}{h_y \circ \pi} = 1$  on the exceptional divisor  $\mathcal{D}$ . Therefore  $I(h_{\lambda}) = I(h_y) = y$ . Moreover the isotropy subgroup of each  $h_{\lambda}$  in  $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$  coincides with the isotropy subgroup of  $h$  and therefore  $\tilde{X}_{h_{\lambda}}$  is constant. This proves the statement.  $\square$

A difference between the equation (4) and the equations in Theorems 1 and 2 in [3] convinced us that the equations in [3] contain a mistake. Here we correct this mistake and formulate a somewhat more general statement than Theorems 1 and 2 in [3]. There we considered two cases: a collection of divisorial valuations and a collection of curve valuations. Here we assume that the collection  $\{\nu_1, \dots, \nu_r\}$  may contain both kinds of valuations on  $\mathcal{O}_{\mathbb{C}^2,0}$ .

Let  $T = (X, \underline{v}, \alpha)$  be a locally finite  $(G, r)$ -set. Let us define its symmetric power  $S^k T$  in the following way. The  $(G, r)$ -set  $S^k T$  is the triple  $(S^k X, \underline{v}^{(k)}, \alpha^{(k)})$  where  $S^k X$  is the  $k$ -th symmetric power of the  $G$ -set  $X$  (with the natural  $G$ -action),  $\underline{v}^{(k)}(\{x_1, \dots, x_k\}) = \sum_{i=1}^k \underline{v}(x_i)$ , where the unordered collection  $\{x_1, \dots, x_k\}$  represents a point in  $S^k X$  and  $\alpha^{(k)}$  is defined as above (for symmetric powers of equipped  $G$ -sets). If the locally-finite  $(G, r)$ -set  $T$  is such that  $\underline{v}^{-1}(0) = 1$  (i.e. a point), then  $ST = \bigsqcup_{k=0}^{\infty} S^k T$  is a locally finite  $(G, r)$ -set as well. The  $(G, r)$ -set  $T_{\Xi}$  defined in [3] has this property. Thus  $ST_{\Xi}$  is defined as an element of  $K_0((G, r) - \text{sets})$ .

**Theorem 2** *One has*

$$P_{\{\nu_i\}}^G = \prod_{\Xi} (ST_{\Xi})^{\chi(\Xi)}.$$

The proof is essentially the one given above.

Theorems 1 and 2 from [3] were used in [4]. Thus from formal point of view one can assume that the results in [4] are not fully proved. However essentially we did not use the equations of Theorems 1 and 2 but only the fact that the knowledge of the Poincaré series  $P_{\{\nu_i\}}^G \in K_0((G, r) - \text{sets})$  permitted to restore the numbers  $\sum_{\Xi: T_{\Xi}=T} \chi(\Xi)$ . In that paper this followed from the unique factorization of the Poincaré series of the form

$$P_{\{\nu_i\}}^G = \prod_T (1 - T)^{-s_T}$$

where the product is over all irreducible elements of the ring  $K_0((G, r) - \text{sets})$  such that  $\underline{v}(x) > 0$  for all  $x \in X$ ,  $s_T \in \mathbb{Z}$ . However, in the same way, one has the unique factorization of the form

$$P_{\{\nu_i\}}^G = \prod_T (ST)^{s_T}$$

where  $ST = 1 + T + S^2 T + \dots$  and the product is over all the irreducible elements of  $K_0((G, r) - \text{sets})$  with the mentioned property. This follows from the fact that the irreducible elements  $T = (X, \underline{v}, \alpha)$  of  $K_0((G, r) - \text{sets})$  with  $\underline{v}(x) > 0$  can be partially ordered in such a way that

- 1) for  $k > 1$ ,  $S^k T$  contains only irreducible components greater than  $T$ ;
- 2) each set of irreducible components has a minimal one.



In this case if  $P_{\{\nu_i\}}^G = 1 + sT_1 + \dots$  where  $T_1$  is a minimal irreducible  $(G, r)$ -set in the decomposition of  $P_{\{\nu_i\}}^G$  with  $s \neq 0$  then  $P_{\{\nu_i\}}^G = (ST_1)^s \cdot Q$  where  $Q = 1 + \dots$  (monomials not smaller than  $T_1$ ). Then one can apply the factorization procedure to  $Q$  and get the result. The required order can be defined, e.g., in the following way. For an irreducible  $(G, r)$ -set  $T = (X, \underline{v}, \alpha)$ , let  $\underline{w}(T) := \sum_{a \in G} \underline{v}(ax)$  for a point  $x \in X$  (one can see that  $\underline{w}(T)$  does not depend on  $x$ ). Then we say that  $T_1 < T_2$  if and only if  $\underline{w}(T_1) < \underline{w}(T_2)$ . One can see that a symmetric power of an irreducible element  $T$  (with  $\underline{w}(x) > 0$ ) contains only components greater than  $T$ .

### 3 Relations with the equivariant monodromy zeta functions

In [6], [7], there were defined two versions of the equivariant monodromy zeta function for a  $G$ -invariant function germ  $f$  as a series from  $(A(G) \otimes \mathbb{Q})[[t]]$  (in [6]) and from  $A(G)[[t]]$  (in [7]). We shall show that in the case when  $(V, 0) = (\mathbb{C}^2, 0)$  (i.e. on the plane), with some exceptions, these monodromy zeta functions can be restored (using a simple algorithm) from the equivariant Poincaré series  $P_{\{\nu_i\}}^G(\underline{t}) \in \hat{A}(G)[[t_1, \dots, t_r]]$  for the set  $\{\nu_i\}$  of the curve valuations defined by the components of the  $G$ -invariant curve  $\{f = 0\}$ .

Let  $(\mathbb{C}^2, 0)$  be endowed with a  $G$ -representation and let  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  be a  $G$ -invariant function germ. Let  $C = \{f = 0\}$  be its zero-level curve. Let  $C = \bigcup_{i=0}^r C_i$  be the representation of  $C$  as the union of irreducible components (among the curves  $C_i$  one can have identical ones), and let  $\nu_i$  be the curve valuation defined by the irreducible curve  $C_i$ . Let  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  be the equivariant Poincaré series of the collection  $\{\nu_i\}$  of valuations.

**Remark.** Assume that one takes one irreducible component from each orbit of the component of the curve  $C$  with the reduced structure, say,  $C'_1, \dots, C'_s$ . The equivariant Poincaré series  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  can be obtained from the equivariant Poincaré series  $P_{\{\nu'_i\}}^G(t'_1, \dots, t'_s)$  by substituting each variable  $t'_j$  by the product of the corresponding variables  $t_i$ .

Let  $\zeta_f^G(t) \in (A(G) \otimes \mathbb{Q})[[t]]$  and  $\tilde{\zeta}_f^G(t) \in A(G)[[t]]$  be the monodromy zeta functions of the  $G$ -invariant germ  $f$  defined in [6] and [7] respectively. One cannot hope to restore the Poincaré series  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  from the series  $\zeta_f^G(t)$  or  $\tilde{\zeta}_f^G(t)$  since, in particular, the Poincaré series is a series in a number of variables and thus is a more fine invariant than the zeta functions. In particular, the monodromy zeta function does not determine the Poincaré series of a non-irreducible plane curve singularity in the non-equivariant situation,

i.e.  $G = (e)$ . (In this case the Poincaré series coincides with the multi-variable Alexander polynomial which can be considered as a multi-variable generalization of the monodromy zeta function. The multi-variable Alexander polynomial, but not the monodromy zeta function, determines the topology of a curve singularity: [13], [12]). Therefore we shall discuss the possibility to restore the equivariant monodromy zeta functions  $\zeta_f^G(t)$  and  $\tilde{\zeta}_f^G(t)$  from the equivariant Poincaré series  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  or from its reduction  $\rho P_{\{\nu_i\}}^G(t_1, \dots, t_r) \in A(G)[[t_1, \dots, t_r]]$ .

One can easily see that

$$\rho P_{\{\nu_i\}}^G(t_1, \dots, t_r) = \prod_{\Xi} (1 - t^{\omega_{\Xi}})^{-\chi(\Xi)[G/H_{\Xi}]} \quad (5)$$

where  $H_{\Xi} = G_x$  for  $x \in p^{-1}(\Xi)$  (since  $\rho[\tilde{X}_{\Xi}] = [G/H_{\Xi}] \in A(G)$ ).

The equivariant zeta functions can be given in terms of a  $G$ -resolution  $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  of the curve  $C$ . Let  $\{\Xi\}$  be the stratification of the space  $\hat{\mathcal{D}} = \mathring{\mathcal{D}}/G$  described in Section 2. For  $x \in \mathring{\mathcal{D}}$ , let

$$n_x = \sum_{i=1}^r \nu_i(h_x) = |\underline{m}_x|. \quad (6)$$

The isotropy subgroup  $G_x$  of the point  $x$  acts on the  $G_x$ -invariant normal slice to  $\mathring{\mathcal{D}}$  at the point  $x$  (in fact the strict transform of the  $G_x$ -invariant curvette  $L_x$  corresponding to  $x$ ). Let  $\hat{G}_x \subset G_x$  be the isotropy subgroup of a point of this slice different from  $x$ .

The integer  $n_x$  and the conjugacy class of the pair  $(G_x, \hat{G}_x)$  are the same for all the points  $x$  from the preimage of a stratum  $\Xi$  with respect to the factorization map  $p : \mathring{\mathcal{D}} \rightarrow \mathring{\mathcal{D}}/G$ . Therefore let us denote them by  $n_{\Xi}$  and  $(H_{\Xi}, \hat{H}_{\Xi})$  respectively.

Then one has ([6], [7]):

$$\begin{aligned} \zeta_f^G(t) &= \prod_{\Xi} (1 - t^{n_{\Xi}})^{-\frac{|\hat{H}_{\Xi}|\chi(\Xi)}{|\hat{H}_{\Xi}|}[G/\hat{H}_{\Xi}]} , \\ \tilde{\zeta}_f^G(t) &= \prod_{\Xi} \left( 1 - t^{n_{\Xi} \frac{|\hat{H}_{\Xi}|}{|\hat{H}_{\Xi}|}} \right)^{-\chi(\Xi)[G/\hat{H}_{\Xi}]} . \end{aligned}$$

Let us first assume that the action of  $G$  on  $\mathbb{C}^2 \setminus \{0\}$  is free. In this case

$\widehat{H}_\Xi = (e)$  for any  $\Xi$ . Therefore one has

$$\begin{aligned}\zeta_f^G(t) &= \prod_{\Xi} (1 - t^{n_\Xi})^{-\frac{\chi(\Xi)}{|H_\Xi|}[G/(e)]} , \\ \widetilde{\zeta}_f^G(t) &= \prod_{\Xi} \left(1 - t^{\frac{n_\Xi}{|H_\Xi|}}\right)^{-\chi(\Xi)[G/(e)]} .\end{aligned}$$

The equations (3) and (6) imply that  $n_\Xi = \frac{|w_\Xi|}{|G|}$ , Therefore the equivariant monodromy zeta functions  $\zeta_f^G(t)$  and  $\widetilde{\zeta}_f^G(t)$  can be restored from the A'Campo type decomposition of the series  $\rho P_{\{\nu_i\}}^G(t, \dots, t)$  in the obvious way. Thus we have the following statement.

**Proposition 1** *If the  $G$ -action on  $\mathbb{C}^2 \setminus \{0\}$  is free, the reduced equivariant Poincaré series  $\rho P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  determines the equivariant zeta functions  $\zeta_f^G(f)$  and  $\widetilde{\zeta}_f^G(f)$ .*

**Remark.** The algorithm to restore the monodromy zeta functions from the reduced equivariant Poincaré series  $\rho P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  includes the factorization of it into the product of the binomials of the form  $(1 - t^{\underline{w}})^{-s_{[H], \underline{w}}[G/H]}$ . Without this factorization an algorithm is not clear.

For an arbitrary faithful action of a finite group  $G$  on  $\mathbb{C}^2$ , the equivariant monodromy zeta functions  $\zeta_f^G(G)$  and  $\widetilde{\zeta}_f^G(G)$  contains some factors of the form  $(1 - t^k)^{-s_{[G/(e)]}}$  and also factors of the form  $(1 - t^k)^{-[G/H]}$  corresponding to lines consisting of points with a non-trivial isotropy subgroup  $H$ . (We shall call these lines *exceptional* ones.) If the representation of  $G$  on  $\mathbb{C}^2$  is known, one knows all these possible factors and the problem to restore the monodromy zeta functions from the equivariant Poincaré series is somewhat simpler. Therefore we shall not assume the representation of  $G$  on  $\mathbb{C}^2$  to be known.

One can see that to make it possible to restore the equivariant zeta functions from the equivariant Poincaré series  $P_{\{\nu_i\}}^G(\underline{t})$  one has to exclude certain cases (cf. with Theorem 3 in [4]).

**Example.** Let us consider two actions of the cyclic group  $\mathbb{Z}_6$  on  $\mathbb{C}^2$  by representations  $\sigma *_1(x, y) = (\sigma^2 x, \sigma y)$  and  $\sigma *_2(x, y) = (\sigma^3 x, \sigma y)$  respectively ( $\sigma = \exp(2\pi i/6)$  is the generator of  $\mathbb{Z}_6$ ) and let  $C$  be the curve  $\{x^6 = 0\}$ . One has  $P_{\{\nu_i\}}^{\mathbb{Z}_6}(\underline{t}) = (1 - t_1 \cdot \dots \cdot t_6)^{-[\mathbb{Z}_6/\mathbb{Z}_6]_\sigma}$ , where  $\sigma$  is the one-dimensional representation defined by  $\sigma * z = \sigma z$ . Moreover the equivariant zeta functions  $\zeta_f^G(t)$  and  $\widetilde{\zeta}_f^G(t)$  are different in these two cases, they are of the form  $(1 - t^k)^{-s_{[\mathbb{Z}_6/\mathbb{Z}_2]}}$  and to  $(1 - t^k)^{-s_{[\mathbb{Z}_6/\mathbb{Z}_3]}}$ ,  $s \neq 0$ , respectively. Thus they are not determined by the equivariant Poincaré series  $P_{\{\nu_i\}}^{\mathbb{Z}_6}(\underline{t})$ .

**Theorem 3** *Let  $\mathbb{C}^2$  be endowed with a faithful action (a representation) of a finite group  $G$  and let  $C = \{f = 0\} = \bigcup_{i=1}^r C_i$ , where some of the components  $C_i$  may coincide, be the zero level curve of a  $G$ -invariant function germ  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ . Assume that the curve  $C$  does not contain a smooth curve invariant with respect to a non-trivial element of  $G$  whose action on  $\mathbb{C}^2$  is not a scalar one. Let  $\{\nu_i\}_{i=1}^r$  be the corresponding collection of valuations. Then the  $G$ -equivariant Poincaré series  $P_{\{\nu_i\}}^G(t_1, \dots, t_r)$  determines the equivariant zeta functions  $\zeta_f^G(t)$  and  $\tilde{\zeta}_f^G(t)$ .*

**Proof.** Let

$$P_{\{\nu_i\}}^G(\underline{t}) = \prod_{\underline{w}, [H], \alpha} (1 - \underline{t}^{\underline{w}})^{-s_{[H], \underline{w}, \alpha} [G/H]_\alpha}$$

be the A'Campo type factorization of the equivariant Poincaré series (here  $[H]$  runs over the conjugacy classes of subgroups of  $G$ ,  $\alpha$  is a one-dimensional representation of  $H$ ). In order to restore the zeta functions  $\zeta_f^G(t)$  and  $\tilde{\zeta}_f^G(t)$  one has to “localize” the factors corresponding to the exceptional lines and to determine the action of the corresponding isotropy subgroups (the subgroups preserving the lines) on them.

First we shall consider the case of an Abelian group  $G$ . This makes the idea of the proof more transparent and permits to describe the general case in a shorter way.

A representation of an Abelian group  $G$  splits into the direct sum of two one-dimensional representations. The action of  $G$  can be not free on some of the corresponding coordinate lines (on non of them, or on one of them, or on both of them). If the action on a coordinate line is not free, this is an exceptional one. According to the assumption of the Theorem these exceptional lines are not contained in the curve  $C$ . In the equation 4 these exceptional lines represent a zero-dimensional strata of the stratification  $\{\Xi\}$  and provide to  $P_{\{\nu_i\}}^G(\underline{t})$  factors of the form  $(1 - \underline{t}^{\underline{w}})^{-[G/G]_\alpha}$ , where  $\alpha$  is a one-dimensional representation of  $G$ . Moreover, this factor has a minimal exponent  $\underline{w}$  among the factors of the form  $(1 - \underline{t}^{\underline{w}'})^{-s_{G, \underline{w}', \alpha} [G/G]_\alpha}$  with positive  $s_{G, \underline{w}', \alpha}$ . This gives the factor corresponding to one of the axis. Let us consider factors of the form  $(1 - \underline{t}^{\underline{w}'})^{-s_{[G/G]_\beta}}$  with  $\beta \neq \alpha$ . If such factors do not exist, the action of  $G$  is a scalar one (i.e. an element  $a \in G$  acts on  $\mathbb{C}^2$  by multiplication by  $\alpha(a)$ ). In this case the action of  $G$  on  $\mathbb{C}^2 \setminus \{0\}$  is free and thus the equivariant zeta functions  $\zeta_f^G(t)$  and  $\tilde{\zeta}_f^G(t)$  are restored from the equivariant Poincaré series as above (in Proposition 1). If factors of the form  $(1 - \underline{t}^{\underline{w}'})^{-s_{[G/G]_\beta}}$  with  $\beta \neq \alpha$  exist, then the minimal among them with  $s$  positive (in fact with  $s = 1$ ) corresponds to the other axis. Moreover, the representation of  $G$  on  $\mathbb{C}^2$  is the direct sum of the one-dimensional representations  $\alpha$  and  $\beta$ . The appearance

of the corresponding factors in the equivariant zeta functions depends on the representations  $\alpha$  and  $\beta$ . (If both of them are exact, the  $G$ -action on  $\mathbb{C}^2 \setminus \{0\}$  is free and the way to restore the equivariant zeta functions is as in Proposition 1.)

In general, to get the equivariant zeta function  $\zeta_f^G(t)$  from the equivariant Poincaré series, one has to substitute the described factors  $(1 - \underline{t}^{\underline{w}_1})^{-[G/G]_\alpha}$  and  $(1 - \underline{t}^{\underline{w}_2})^{-[G/G]_\beta}$  by

$$\left(1 - t^{\frac{|\underline{w}_1|}{|G|}}\right)^{-\frac{|\text{Ker } \beta|}{|G|}[G/\text{Ker } \beta]} \quad \text{and} \quad \left(1 - t^{\frac{|\underline{w}_2|}{|G|}}\right)^{-\frac{|\text{Ker } \alpha|}{|G|}[G/\text{Ker } \alpha]}$$

respectively. (Note that in some sense the representations  $\alpha$  and  $\beta$  exchange their positions.) All other factors have to be modified as in Proposition 1.

To get the equivariant zeta function  $\tilde{\zeta}_f^G(t)$ , one has to substitute these two factors by

$$\left(1 - t^{\frac{|\underline{w}_1| |\text{Ker } \beta|}{|G|^2}}\right)^{-[G/\text{Ker } \beta]} \quad \text{and} \quad \left(1 - t^{\frac{|\underline{w}_2| |\text{Ker } \alpha|}{|G|^2}}\right)^{-[G/\text{Ker } \alpha]}$$

respectively.

Now let  $G$  be an arbitrary finite group acting faithfully on  $\mathbb{C}^2$ . An exceptional line is the set of fixed points of an Abelian subgroup  $H \subset G$ ,  $H \neq (e)$ . The elements of the normalizer  $N_G(H)$  preserve the line. All other elements of  $G$  send the line to other exceptional ones. The normalizer  $N_G(H)$  is an Abelian group. Its representation on  $\mathbb{C}^2$  splits into two different one-dimensional representations. (If these two representations coincide, the action of  $N_G(H)$  is a scalar one and thus  $N_G(H)$ , and therefore  $H$ , act freely on  $\mathbb{C}^2 \setminus \{0\}$ .) The corresponding coordinate lines provide into the equation (4) the factors of the form  $(1 - \underline{t}^{\underline{w}_1})^{-[G/N_G(H)]_\alpha}$  and  $(1 - \underline{t}^{\underline{w}_2})^{-[G/N_G(H)]_\beta}$  which are localized in the same way as above (for an Abelian  $G$ ). This means that one of them has a minimal exponent  $\underline{w}'$  among the factors of the form  $(1 - \underline{t}^{\underline{w}'})^{-s[G/N_G(H)]_\gamma}$  with positive  $s$  and the other one has a minimal exponent  $\underline{w}'$  among the similar factors with a different representation.

Now, to get the equivariant zeta function  $\zeta_f^G(t)$  from the equivariant Poincaré series, one has to substitute the obtained factors  $(1 - \underline{t}^{\underline{w}_1})^{-[G/N_G(H)]_\alpha}$  and  $(1 - \underline{t}^{\underline{w}_2})^{-[G/N_G(H)]_\beta}$  by

$$\left(1 - t^{\frac{|\underline{w}_1|}{|G|}}\right)^{-\frac{|\text{Ker } \beta|}{|N_G(H)|}[G/\text{Ker } \beta]} \quad \text{and} \quad \left(1 - t^{\frac{|\underline{w}_2|}{|G|}}\right)^{-\frac{|\text{Ker } \alpha|}{|N_G(H)|}[G/\text{Ker } \alpha]}$$

respectively.

To get the equivariant zeta function  $\tilde{\zeta}_f^G(t)$ , one has to substitute them by

$$\left(1 - t^{\frac{|w_1||\text{Ker } \beta|}{|G||N_G(H)|}}\right)^{-[G/\text{Ker } \beta]} \quad \text{and} \quad \left(1 - t^{\frac{|w_2||\text{Ker } \alpha|}{|G||N_G(H)|}}\right)^{-[G/\text{Ker } \alpha]}$$

respectively.  $\square$

## References

- [1] Campillo A., Delgado F., Gusein-Zade S.M. The Alexander polynomial of a plane curve singularity and integrals with respect to the Euler characteristic. *Internat. J. Math.*, v.14, no.1, 47–54 (2003).
- [2] Campillo A., Delgado F., Gusein-Zade S.M. On Poincaré series of filtrations on equivariant functions of two variables. *Mosc. Math. J.*, v.7, no.2, 243–255 (2007).
- [3] Campillo A., Delgado F., Gusein-Zade S.M. Equivariant Poincaré series of filtrations. *Rev. Mat. Complut.*, v.26, 241–251 (2013).
- [4] Campillo A., Delgado F., Gusein-Zade S.M. Equivariant Poincaré series of filtrations and topology. *Ark. Mat.*, v.52, 43–59 (2014).
- [5] Campillo A., Delgado F., Kiyek K. Gorenstein property and symmetry for one-dimensional local Cohen–Macaulay rings. *Manuscripta Mathematica*, v.83, no.3–4, 405–423 (1994).
- [6] Gusein-Zade S.M., Luengo I., Melle-Hernández A. An equivariant version of the monodromy zeta function. In: *Geometry, Topology, and Mathematical Physics: S. P. Novikov’s Seminar: 2006–2007* (V.M.Buchstaber and I.M.Krichever, eds.), AMS, 2008 (American Mathematical Society Translations: Series 2, vol.224), pp. 139–146.
- [7] Gusein-Zade S.M., Luengo I., Melle-Hernández A. On an equivariant version of the zeta function of a transformation. arXiv: 1203:3344.
- [8] Knutson D.  $\lambda$ -rings and the representation theory of the symmetric group. *Lecture Notes in Mathematics*, vol.308. Springer-Verlag, Berlin-New York, 1973.
- [9] Stanley R.P. *Enumerative combinatorics. Vol. 2.* Cambridge Studies in Advanced Mathematics **62**. 1999.

- [10] Némethi, A. Poincaré series associated with surface singularities. Singularities I, Contemp. Math., v.474, 271–297 (2008).
- [11] tom Dieck T. Transformation groups and representation theory. Lecture Notes in Mathematics, vol.**766**. Springer, Berlin, 1979.
- [12] Wall C.T.C. Singular points of plane curves. London Mathematical Society Student Texts, vol.**63**. Cambridge University Press, Cambridge, 2004.
- [13] Yamamoto M. Classification of isolated algebraic singularities by their Alexander polynomials. Topology, v.23, no.3, 277–287 (1984).

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